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LETTER TO THE EDITOR

# New class of function-dependent deformations of creation and annihilation operators

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**Abstract.** We give formulae for point spectra of Hermitian operators built from creation and annihilation type operators satisfying general algebraic relation which we call $(g, f, F)$ equivalence. By choosing the parameters in a specific way the description of the quantum $q$-oscillator is given in a model built from the $q$-derivative and dilatation operator. We show the explicit form of eigenfunctions of the model from $L^2(R)$. Then we construct the transformation between our model and $Q$-oscillator with arbitrary $Q$ real.

The aim of our letter is to propose a certain method of constructing the point spectrum and eigenfunctions for the Hermitian operator built from creation and annihilation type operators. The technique is well known for a less general case—first developed for classical oscillators and then for models with a special type of interaction described by shape-invariant potentials. The number of such models was studied in the literature [1-8] and afterwards the theorem about their spectra was proved by Arai [9].

However, we cannot apply Arai's method to the models with $q$-deformed commutation relations. We present here a generalization of the Arai theorem. In the second part of our paper we shall give a realization of $q$-oscillator representation built from the $q$-derivative and acting on the $L^2(R)$ space of real functions.

Let us start with a definition of $(g, f, F)$ equivalence of two families of operators.

*Definition.* $\{B_\lambda\}_{\lambda\in\Lambda}$ and $\{C_\lambda\}_{\lambda\in\Lambda}$ are $(g, f, F)$ equivalent iff

$$B(\lambda) = g(f(\lambda))C(f(\lambda)) + F(f(\lambda)) \tag{1}$$

where

$$f: \Lambda \to \Lambda \qquad g: \Lambda \to R \qquad F: \Lambda \to R.$$

The novelty of this definition lies in including function $g$ in our formula. It was equal to 1 in all the previous works. Having this definition we can state the following proposition.

*Proposition.* Let $A(c)A^+(c)$ and $A^+(c)A(c)$ be $(g, f, F)$ equivalent families of operators. Let further the following normalizable states exist

$$\phi_n \in \operatorname{Ker} A(f^n(c)) \qquad \dim \operatorname{Ker} A(f^n(c)) = d_n \qquad \langle\phi_n|\phi_n\rangle < \infty.$$

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Then we can construct the eigenfunctions of $A^+(c)A(c)$ as

$$|n\rangle = A^+(c)A^+(f(c))\dots A^+(f^{n-1}(c))\phi_n \tag{2a}$$

$$A^+(c)A(c)|n\rangle = \sigma_n|n\rangle \tag{2b}$$

$$\sigma_n = \sum_{l=1}^{n} F(f^l(c)) \prod_{m=1}^{l-1} g(f^m(c)) \qquad n=1,2,\dots \tag{2c}$$

and dimension of eigenspace for value $\sigma_n$ is equal to $d_n$.

The validity of formulae (2) can be easily shown—the proof will be, together with a lemma about the normalization of eigenfunctions, the subject of our next paper.

The quasi-commutation relation (1) is a generalization of relations occurring in mentioned models with shape-invariant potentials and on the other hand it generalizes also the standard commutation formulae for the quantum $q$-oscillator. We consider now a model fulfilling the assumptions of our proposition.

We shall deal with real functions normalizable in the norm given by a scalar product

$$\langle\omega|\rho\rangle = \int_0^\infty \omega\rho\, \mathrm{d}\mu(y_0) = \sum_{n=-\infty}^{\infty} q^{2n}(q^2-1)y_0\omega(q^{2n}y_0)\rho(q^{2n}y_0). \tag{3}$$

Let us recall a few properties of the $q$-derivative and dilatation operator:

$$\begin{aligned} &\zeta_a\omega(x) := \omega(ax) \qquad \partial_q\omega(x) := (\zeta_{q^2}-1)\omega(x)/(q^2-1)x \\ &\partial_q^+ = -\frac{1}{q^2}\zeta_{1/q^2}\partial_q \qquad \partial_q\zeta_a = a\zeta_a\partial_q \qquad (\zeta_{q^{2l}})^+ = q^{-2l}\zeta_{1/q^{2l}}. \end{aligned} \tag{4}$$

Our annihilation and creation operators are as follows

$$A^+(c) = \zeta_a\partial_q + c\zeta_b \qquad A(c) = -\frac{1}{a^2q^2}\zeta_{1/aq^2}\partial_q + \frac{c}{b}\zeta_{1/b}. \tag{5}$$

The bilinear operators $A(c)A^+(c)$ and $A^+(c)A(c)$

$$\begin{aligned} A^+A(c) &= -\frac{1}{a^3q^4}\zeta_{1/q^2}\partial_q\partial_q + \frac{c}{b^2}\zeta_{a/b}\partial_q - \frac{c}{a^2q^2}\zeta_{b/aq^2}\partial_q + \frac{c^2}{b} \\ AA^+(c) &= -\frac{1}{aq^2}\zeta_{1/q^2}\partial_q\partial_q + \frac{c}{b}\zeta_{a/b}\partial_q - \frac{cb}{a^2q^2}\zeta_{b/aq^2}\partial_q + \frac{c^2}{b} \end{aligned} \tag{6}$$

satisfy the condition of $(g, f, F)$ equivalence (1) with functions specified below

$$\begin{aligned} &g(f(c)) = a^2q^2 \qquad f(c) = c\frac{b}{a^2q^2} \\ &F(f(c)) = [(f(c))]^2\frac{a^2q^2}{b}\left(\left(\frac{aq}{b}\right)^2 - 1\right). \end{aligned} \tag{7}$$

We should now build a string of $L^2(R)$ vacuums for $q$ real and $q<1$. The finiteness of their norm can be shown by application of the d'Alembert criterion for convergence of power series using the properties of solutions of the following functional equation:

$$A(f^n(c))\phi_n = \zeta_{1/aq^2}\left[-\frac{1}{a^2q^2}\partial_q + \frac{c}{aq^2}\left(\frac{1}{a}\right)^n\right]\phi_n = -\frac{1}{a^2q^2}\zeta_{1/aq^2}[\partial_q - a^{-n+1}c]\phi_n = 0 \tag{8}$$

where we have chosen

$$aq^2 = b.$$

The explicit form of solution is described by the power series

$$\phi_n = \exp_q(a^{-n+1}cx) = 1 + \sum_{k=1}^{\infty} \frac{(a^{-n+1}cx)^k}{\prod_{l=1}^{k} [l]_q^c} \tag{9}$$

which is convergent when

$$0 \leqslant x < -a^{n-1}/c(q^2-1) = x_0^n \tag{9a}$$

and the notation

$$[n]_q^c = (q^{2n}-1)/(q^2-1)$$

was used. The calculated interval of convergence (9$a$) implies that so far we have defined our vacuum functions for $x \in [0, x_0^n)$ only. In our simple example we can assume the following definition

$$\phi_n = \begin{cases} 0 & x = 0 \\ \exp_q(a^{-n+1}cx) & x \in [0, y_0) \\ 0 & x \in [y_0, \infty) \end{cases} \tag{9b}$$

where $y_0 \leqslant x_0^n$, $n \in N$.

For such a definition of vacuums we must have $a = 1$. Only then do the functions $\phi_n$ defined on $[0, \infty)$ fulfil equation (8).

Note that the solutions (9) have the following property which results from (8) and is crucial in the proof of finiteness of their norm:

$$\begin{aligned} &\phi_n(q^2 q^{2k-2}x) = (1 + (q^2-1)q^{2k-2}xa^{-n+1}c)\phi_n(q^{2k-2}x) \qquad k \in N \\ &\phi_n(q^2 q^{-2k}x) = (1 + (q^2-1)q^{-2k}xa^{-n+1}c)\phi_n(q^{-2k}x) \qquad k \in N. \end{aligned} \tag{10}$$

Each of the vacuums $\phi_n$ (which here are in fact identical) gives the eigenstate

$$|n\rangle = A^+(c)A^+(f(c)) \ldots A^+(f^{n-1}(c))\phi_n \tag{11}$$

corresponding to the eigenvalue $\sigma_n$ of the operator $A^+(c)A(c)$

$$\sigma_n = \frac{c^2}{a}(q^{-2}-1)[n]_{b/aq}^c. \tag{12}$$

Notice that the assumed choice of parameters

$$b = aq^2 \qquad \text{and} \qquad a = 1$$

reduces our $(g, f, F)$ equivalence condition to the $q$-oscillator commutation relation:

$$AA^+ = q^2A^+A + c^2(q^{-2}-1). \tag{13}$$

The constructed model is correct for $q < 1$. Nevertheless we can transform our creation and annihilation operators so as to obtain a representation for arbitrary $Q$. Similar transformations between different types of independent oscillators and their covariant form are known [10, 11]. The proposed transformation acts between oscillators with different parameters and is described by the formulae

$$a = A\chi(A^+A) \qquad a^+ = \chi(A^+A)A^+. \tag{14}$$

The states of the new Fock space are proportional to the previous ones

$$|n) = (a^+)^n\phi = [\chi(A^+A)A^+]^n\phi = \chi(\sigma_1)\chi(\sigma_2)\ldots\chi(\sigma_n)|n\rangle. \tag{15}$$

Let us notice the explicit form of the square of transformation

$$\chi^2(A^+A)=Q^2\frac{(1-(1-q^2)A^+A/F)^{-\log_q Q}-(1-(1-q^2)A^+A/F)^{\log_q Q}}{(1-Q^4)A^+A} \tag{16}$$

where

$$F=c^2(q^{-2}-1).$$

The new Hamiltonian of the $Q$-oscillator has the following form:

$$H=a^+a=Q^2(1-Q^4)^{-1}\sum_{n=1}^{\infty}(-1)^n\left\{\begin{bmatrix}-\log_q Q\\ n\end{bmatrix}-\begin{bmatrix}\log_q Q\\ n\end{bmatrix}\right\}\frac{(1-q^2)^n(A^+A)^n}{n!F^n} \tag{17}$$

where

$$\begin{bmatrix}\alpha\\ n\end{bmatrix}=\alpha(\alpha-1)\dots(\alpha-n+1)$$

and the spectrum

$$a^+a|n)=Q^2(Q^{-2n-2}-Q^{2n+2})/(1-Q^4)|n)=[n+1]_{Q^2}|n). \tag{18}$$

Note a few problems which are implied by our proposition. It seems that this technique can be applied for models where more oscillators are included. One should start then with their independent form and after transformation the spectra of the covariant form of Hamiltonian should be obtained. It is also an interesting point whether the method given in (2) could be applied in calculating spectra of Casimir-type operators in quantum algebras. As regards the transformation proposed for writing down the $Q$-oscillator model, the question arises whether the considered model for the $q$-boson oscillator can be transformed to the fermion one.

The problems mentioned are still under investigation and the results will be published in a subsequent paper.

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